On Modules Over Group Rings

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Noncommutative Rings and Their Applications

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Definition

Let M be a right module over a ring R and let G be a group. Let MG be the set of all formal finite sums of the form $\sum_{g\in G} m_g g$ where $m_g \in M$. For $\sum m_g g, \sum_{g\in G} n_g g \in MG$ and $\sum r_g g \in RG$, by writing $\sum m_g g = \sum n_g g$ we mean that $m_g = n_g$ for all $g \in G$. The sum and the scalar product are defined as follows:

$$\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G} (m_g + n_g) g$$

$$\Big(\sum_{g\in G} m_g g\Big)\Big(\sum_{g\in G} r_g g\Big) = \sum_{g\in G} k_g g, \text{ where } k_g = \sum_{hh'=g} m_h r_{h'} .$$

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Example 2

1. *If* $M = R$ *is the ring itself, then* MG *coincides with the group ring* RG.
2. *If* $M = I$ *is a right ideal of* R*, then* $MG = IG$ *is a right ideal of* RG.
3. *If* G *is the infinite cyclic group, then* $RG = R[x^{-1}, x]$ *is the Laurent polynomial ring and* $MG = M[x^{-1}, x]$ *is the Laurent polynomial module over* $R[x^{-1}, x]$.

Motivated questions

- Study module properties of the module $(MG)_{RG}$, aiming to extend the known results on ring properties of RG.
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The augmentation map

With $m \in M$ being identified with $m \cdot 1 \in MG$ (where 1 is the identity of G), M is an R-submodule of MG. The map

$$\epsilon_M : MG \to M, \ \sum m_g g \mapsto \sum m_g,$$

is an R-homomorphism. The kernel of ϵ_M is denoted by $\Delta(MG)$. Thus, $\epsilon := \epsilon_R : RG \to R$ is the usual *augmentation map*.

Lemma 3

Let MG be the group module of G by M over RG. Then:

1. *For any $x \in MG$ and any $\alpha \in RG$, $\epsilon_M(x\alpha) = \epsilon_M(x)\epsilon(\alpha)$. In particular, ϵ_M is an R-homomorphism.*
2. *$\Delta(MG) = \left\{ \sum m_g(g-1) : g \in G, m_g \in M \right\}$ is an RG-submodule of MG.*

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Basic facts

1. *Let $\{M_i : i \in I\}$ be a family of right R-modules and let G be a group. Then*
$$\left(\left(\bigoplus_{i \in I} M_i \right) G \right)_{RG} \cong \left(\bigoplus_{i \in I} M_i G \right)_{RG}.$$
2. *An R-module M_R is projective if and only if $(MG)_{RG}$ is projective.*
3. *If $Y \cap \Delta(MG) = 0$ for some nonzero submodule Y of $(MG)_{RG}$, then G is a finite group.*

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Semisimple modules

A module M_R is called *semisimple* if M_R is a direct sum of simple R-modules, or equivalently if every submodule of M_R is a direct summand. A ring R is semisimple Artinian if and only if R is a semisimple right (or left) module over R.

Maschke's Theorem

Let G be a finite group. Then the group ring RG is a semisimple Artinian ring if and only if R is a semisimple Artinian ring and $|G|$ is invertible in R.

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The generalized Maschke's Theorem

The group ring RG is a semisimple Artinian ring if and only if R is a semisimple Artinian ring and G is a finite group whose order is invertible in R. [I.G. Cornell, 1963]

Theorem 1 (Kosan-Lee-Z)

Let M_R be a nonzero module and let G be a group. TFAE:

1. *MG is a semisimple module over RG.*
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Injective modules

A module M_R is called *injective* if, for any module Y_R and any R-submodule X of Y, every R-homomorphism $X \to M$ can be extended to an R-homomorphism $Y \to M$. By the Baer Criterion, M_R is injective if and only if, for any right ideal I of R, every R-homomorphism $I \to M$ can be extended to an R-homomorphism $R_R \to M$. The ring R is called right *self-injective* if the module R_R is injective.

I.G. Connell, 1963

For a finite group G, the group ring RG is right self-injective if and only if R is right self-injective.

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Theorem (Connell - Renault)

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Theorem 2 (Kosan-Lee-Z)

Let M_R be a nonzero module and let G be a group. Then $(MG)_{RG}$ is injective if and only if M_R is injective and G is finite.

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FP-injective modules

A module M_R is called *FP-injective* if, for any finitely generated submodule K of a free R-module F, every R-homomorphism $K \to M$ can be extended to an R-homomorphism $F \to M$. A ring R is called *right FP-injective* if R_R is FP-injective.

G.A. Garkusha, 1999

The group ring RG is right FP-injective if and only if R is right FP-injective and G is locally finite.

Theorem 3 (Kosan-Lee-Z)

Let M_R be a nonzero module and let G be a group. Then $(MG)_{RG}$ is FP-injective if and only if M_R is FP-injective and G is locally finite.

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Regular modules

A ring R is called von Neumann regular if, for each $a \in R$, $a = aba$ for some $b \in R$.

Auslander, 1957; Connell, 1963; McLaughlin, 1958

The group ring RG is von Neumann regular if and only if R is von Neumann regular, G is locally finite, and the order of any finite subgroup of G is invertible in R.

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Following [Zelmanowitz, 1972], a module M_R is called *regular* if for any $m \in M$, $m = mf(m)$ for some $f \in \mathrm{Hom}_R(M, R)$. Thus, a ring R is von Neumann regular if and only if R_R is a regular module.

Theorem 4 (Kosan-Lee-Z)

Let M_R be a nonzero module and let G be a group. TFAE:

1. *MG is a regular module over RG.*
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Questions

- The group ring RG is right Artinian if and only if R is right Artinian and G is finite. [G. Connell, 1963]
- The group ring RG is perfect if and only if R is perfect and G is finite. [G. Renault 1971; S.M. Woods, 1971]
- The group ring RG is right pure-injective if and only if R is right pure-injective and G is finite [W. Zimmermann, 1982]

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Characterize when $(MG)_{RG}$ is Artinin (resp., perfect or pure-injective).

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Thank You!

Recall that a right module M over ring R is called algebraically compact, also called pure-injective, if each system of equations $\sum_{i\in I} X_i a_{ij} = m_j$, $j \in J$, with $a_{ij} \in R$ and $m_j \in M$, which is finitely solvable in M^I, has a solution.